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The coherent state on $SU_q(2)$ homogeneous space

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Abstract
The generalized coherent states for quantum groups introduced by Jurčo and Šťovíček are studied for the simplest example $SU_q(2)$ in full detail. It is shown that the normalized $SU_q(2)$ coherent states enjoy the property of completeness, and allow a resolution of the unity. This feature is expected to play a key role in the application of these coherent states in physical models. The homogeneous space of $SU_q(2)$, i.e. the q-sphere of Podleś, is reproduced in complex coordinates by using the coherent states. Differential calculus in the complex form on the homogeneous space is developed. The high spin limit of the $SU_q(2)$ coherent states is also discussed.

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1. Introduction

Generalization of a boson coherent state based on a group theoretical viewpoint allows us to define coherent states for arbitrary Lie groups [1, 2]. Extensive study of such generalized coherent states has revealed physical and mathematical richness of the notion. Physical applications [3–5] of the generalized coherent states are found in many instances such as quantum optics, semiclassical quantization of systems with spin degrees of freedom, transition from a pure state to mixed state dynamics during a nuclear collision, and so on. Mathematically, they provide [3–5] a natural framework to study the geometric structure of the homogeneous space of the group under consideration.

On the other hand, the advent of quantum groups and quantum algebras [6–12] popularized deformations of Lie groups and Lie algebras in theoretical physics and mathematics. It is known that many properties of Lie groups and Lie algebras have their deformed counterparts. Especially, similarity in representation theories is remarkable. This motivates many researchers to consider the coherent state for quantum algebras. Here we mention the pioneering works [13, 14], and attempts [15–19] to construct the generalized coherent states of the type [1, 2] in the context of quantum groups. The generalized coherent state is a vector in the representation space of the Lie group, and it is regarded as a function on homogeneous space with respect to a fiducial vector. For many Lie groups, such coherent states are obtained employing the duality between the group and its algebra. To see this, it may be enough to recall that the representation of a Lie group used in the construction of coherent states is obtained by the exponential mapping from a Lie algebra to the corresponding Lie group. Therefore generalized coherent states for quantum groups should embody the duality of a quantum group and its quantum algebra. Such a definition was introduced by Jurčo and Šťovíček [20]. They developed a general theory of the quantum group version of generalized coherent states starting with the q-analogue of the Iwasawa decomposition and using the quantum double technique. The coherent states in [20] reflect full Hopf algebra structure of quantum groups so that they may be the most plausible generalization of the coherent states of [1, 2]. Nevertheless, to our knowledge, few works have been done on the coherent states for the quantum groups till today.

In the present work, taking the simplest quantum group $SU_q(2)$ as an example, we study the coherent states defined in [20] in full detail. Our construction of the coherent states is basically same as in [20], but there is a slight difference. A central object for the coherent state construction is the universal $\mathcal{T}$-matrix, referred to as the canonical element in [20]. Instead of using the Iwasawa decomposition as done in [20] we derive the universal $\mathcal{T}$-matrix via the method of Fronsdal and Galindo [21], since it makes clear that the universal $\mathcal{T}$-matrix is nothing but a quantum analogue of the exponential mapping from a Lie algebra to a Lie group. Anticipating that the $SU_q(2)$ coherent states may have applications in developing field theories on noncommutative spaces, we put emphasis on the resolution of the unit operator and the complex description of the $SU_q(2)$ homogeneous space, i.e., the Podleś [22] q-sphere. Noncommutativity of the variables parametrizing the coherent states for quantum groups was discussed earlier, for instance, in [23].

The plan of this paper is as follows: the next preliminary section starts reviewing the quantum algebra $U_q[su(2)]$, and then the quantum group $SU_q(2)$ is introduced as a dual algebra in the sense of [21]. The $*$-structure of $SU_q(2)$ is studied in some detail emphasizing the $*$-conjugation properties of the noncommutative generators of the function algebra. We construct the $SU_q(2)$ coherent states and study their properties in section 3. Especially, the resolution of unity is proved in an algebraic setting. In section 4 we show that the $SU_q(2)$ coherent states naturally give the complex description of q-sphere of Podleś. Differential calculus on the q-sphere in complex description is explicitly provided. The Fock–Bargmann type representations of $U_q[su(2)]$ are considered in section 5 as an application of the resolution of unity. High spin limit of the $SU_q(2)$ coherent state is investigated in section 6. It is shown that the limit gives a contraction to the coherent state of a quantum Heisenberg group. We devote section 7 to concluding remarks.

2. Duality and $*$-structure

The well-known [6–9] quantum algebra $\mathcal{U} = U_q[su(2)]$ endowed with a Hopf $*$-structure is generated by three elements $J_\pm$, J_0 subject to the relations

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = [2J_0]_q, \tag{2.1}$$

where the q-deformed construct $[\mathcal{X}]$ reads

$$[\mathcal{X}]_q = \frac{q^{\mathcal{X}} - q^{-\mathcal{X}}}{q - q^{-1}}. \tag{2.2}$$

The coproduct Δ, the counit ϵ and the antipode S maps given below

$$\Delta(J_0) = J_0 \otimes 1 + 1 \otimes J_0, \qquad \Delta(J_\pm) = J_\pm \otimes q^{J_0} + q^{-J_0} \otimes J_\pm, \tag{2.3}$$

$$\epsilon(J_0) = \epsilon(J_\pm) = 0, \tag{2.4}$$

$$S(J_0) = -J_0, \qquad S(J_\pm) = -q^{\pm 1} J_\pm, \tag{2.5}$$

together with the $*$-involution for $q \in \mathbb{R}$

$$J_+^* = J_-, \qquad J_-^* = J_+, \qquad J_0^* = J_0 \tag{2.6}$$

satisfy the axioms of the Hopf $*$-algebra. A set of monomials

$$E_{k\ell m} = J_+^k J_0^\ell J_-^m, \qquad k, \ell, m \in \mathbb{Z}_{\geqslant 0} \tag{2.7}$$

provide the basis of the universal enveloping algebra $\mathcal{U}$.

The algebra dual to $\mathcal{U}$ is the quantum group $\mathcal{A} = SU_q(2)$. One can determine the basis elements of the algebra $\mathcal{A}$ and their Hopf structure *à la* Fronsdal and Galindo [21] where they started from two parameter deformation of $SL(2)$, and reached its dual algebra $\mathcal{U}$. Reversing their construction, we derive the basis vectors $e^{k\ell m}$ of the quantum group $\mathcal{A}$ by employing their dual relations with the known basis set $E_{k\ell m}$ of $\mathcal{U}$

$$\langle e^{k\ell m}, E_{k'\ell' m'} \rangle = \delta^k_{k'} \delta^\ell_{\ell'} \delta^m_{m'}. \tag{2.8}$$

The two sets of structure constants defined below express the duality of $\mathcal{U}$ and $\mathcal{A}$ as follows:

$$E_{k\ell m} E_{k'\ell' m'} = \sum_{abc} f_{k\ell m\, k'\ell' m'}^{\quad abc} E_{abc}, \qquad \Delta(E_{k\ell m}) = \sum_{\substack{abc \\ a'b'c'}} g^{abc\, a'b'c'}_{\ k\ell m} E_{abc} \otimes E_{a'b'c'}, \tag{2.9}$$

$$e^{k\ell m} e^{k'\ell' m'} = \sum_{abc} g^{k\ell m\, k'\ell' m'}_{\quad abc} e^{abc}, \qquad \Delta(e^{k\ell m}) = \sum_{\substack{abc \\ a'b'c'}} f_{abc\, a'b'c'}^{\ k\ell m} e^{abc} \otimes e^{a'b'c'}. \tag{2.10}$$

Denoting the generators of $\mathcal{A}$ as

$$e^{100} = x, \qquad e^{010} = z, \qquad e^{001} = y,$$

we use (2.1) and (2.3) to read off the structure constants in (2.9) and thereby determine the algebraic relations among the generators

$$[x, y] = 0, \qquad [x, z] = 2 \ln q x, \qquad [y, z] = 2 \ln q y. \tag{2.11}$$

These are one parameter reductions of the relations in [21]. The full set of the dual basis $e^{k\ell m}$ may be derived using the following recipe. We start by listing the coproduct structure

$$\begin{aligned} \Delta(E_{k\ell m}) = \sum_{k'\ell' m'} \begin{bmatrix} k \\ k' \end{bmatrix}_q \begin{pmatrix} \ell \\ \ell' \end{pmatrix} \begin{bmatrix} m \\ m' \end{bmatrix}_q & J_+^{k-k'} q^{-k' J_0} J_0^{\ell-\ell'} q^{-m' J_0} J_-^{m-m'} \\ & \otimes J_+^{k'} q^{(k-k')J_0} J_0^{\ell'} q^{(m-m')J_0} J_-^{m'}, \end{aligned} \tag{2.12}$$

where the classical and the q-binomial coefficients read, respectively

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{2.13}$$

Employing (2.12) the structure constants $g^{abc\, a'b'c'}_{\ k\ell m}$ defined in the second equation in (2.9) may be read explicitly. The duality requirement ensures that these constants compose the algebraic multiplication relations of the basis set $e^{k\ell m}$ in (2.10). Using an inductive procedure we now construct the basis set

$$e^{k\ell m} = \frac{x^k}{[k]_q!} \frac{(z-(k-m)\ln q)^\ell}{\ell!} \frac{y^m}{[m]_q!}. \tag{2.14}$$

The universal $\mathcal{T}$-matrix that caps the duality structure of the Hopf algebras $\mathcal{U}$ and $\mathcal{A}$ is defined by $\mathcal{T} = \sum_{k\ell m} e^{k\ell m} \otimes E_{k\ell m}$, and its closed form expression may be obtained by substituting the dually related basis sets given in (2.7) and (2.14), respectively

$$\mathcal{T} = \left(\sum_{k=0}^{\infty} \frac{(x \otimes J_+ q^{-J_0})^k}{(k)_{q^{-2}}!} \right) e^{z \otimes J_0} \left(\sum_{m=0}^{\infty} \frac{(y \otimes q^{J_0} J_-)^m}{(m)_{q^2}!} \right), \tag{2.15}$$

where

$$(n)_q = \frac{1-q^n}{1-q}. \tag{2.16}$$

We emphasize that the classical $q \to 1$ limit of the universal $\mathcal{T}$-matrix (2.15) yields the usual exponential mapping from the Lie algebra $su(2)$ to the Lie group $SU(2)$.

The universal $\mathcal{T}$-matrix (2.15) reproduces the standard matrix expression of $\mathcal{A}$ at the fundamental representation (π) of $\mathcal{U}$

$$\pi(J_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \pi(J_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \pi(J_0) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}. \tag{2.17}$$

Denoting the 2×2 matrix by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv (\mathrm{id} \otimes \pi)(\mathcal{T}) = \begin{pmatrix} \mathrm{e}^{z/2} + x\,\mathrm{e}^{-z/2} y & q^{1/2} x\,\mathrm{e}^{-z/2} \\ q^{-1/2}\,\mathrm{e}^{-z/2} y & \mathrm{e}^{-z/2} \end{pmatrix}, \tag{2.18}$$

one verifies that the elements a, b, c, d satisfy the familiar defining relations of the quantum group $SL_q(2, \mathbb{C})$

$$\begin{aligned} &ab = q^{-1}ba, && ac = q^{-1}ca, && bd = q^{-1}db \\ &cd = q^{-1}dc, && bc = cb, && [a,d] = (q^{-1}-q)bc, \\ &ad - q^{-1}bc = 1. \end{aligned} \tag{2.19}$$

In other words, (2.18) is nothing but the Gauss decomposition of $SL_q(2, \mathbb{C})$ given in [21]. We remark that a, d are invertible, but b, c need not be so. Inverting (2.18), x, y, z are expressed in terms of $SL_q(2, \mathbb{C})$ matrix entries

$$x = q^{-1/2} b d^{-1}, \qquad y = q^{1/2} d^{-1} c, \qquad \mathrm{e}^{z/2} = d^{-1}. \tag{2.20}$$

The quantum group $\mathcal{A}$ is a real form of $SL_q(2, \mathbb{C})$. As is well known, the real form $\mathcal{A}$ is defined by the $*$-involution [12]

$$a^* = d, \qquad b^* = -q^{-1}c, \qquad c^* = -qb, \qquad d^* = a. \tag{2.21}$$

The $*$-involution map of the generators of $\mathcal{A}$ obtained via (2.20) and (2.21)

$$x^* = -q^{-1/2} c a^{-1}, \qquad \mathrm{e}^{z^*/2} = a^{-1}, \qquad y^* = -q^{1/2} a^{-1} b \tag{2.22}$$

maintains a close kinship to the antipode

$$S(x) = y^*, \qquad S(y) = x^*, \qquad S(z) = z^*. \tag{2.23}$$

Introducing the element

$$\zeta = -qx\,\mathrm{e}^{-z} y = -qbc, \tag{2.24}$$

the $*$-map (2.22) is rewritten as

$$x^* = -\frac{1}{1-\zeta} y\,\mathrm{e}^{-z}, \qquad \mathrm{e}^{z^*/2} = \frac{1}{1-\zeta}\,\mathrm{e}^{-z/2}, \qquad y^* = -\frac{1}{1-\zeta}\,\mathrm{e}^{-z} x. \tag{2.25}$$

One immediately notes that the element ζ is real: $\zeta^* = \zeta$. It is observed that x, y, z and their $*$-involutions do not commute. For instance, x and x^* satisfy the relations

$$xx^* = \frac{q^{-2}x^*x}{1+(q-q^{-1})x^*x}, \qquad x^*x = \frac{q^2xx^*}{1-q^2(q-q^{-1})xx^*}. \tag{2.26}$$

For j being a non-negative integer or a positive half-integer the following identities may be proved by induction:

$$\mathrm{e}^{jz^*}\,\mathrm{e}^{jz} = \frac{1}{(\zeta;q^2)_{2j}}, \qquad \mathrm{e}^{jz}\,\mathrm{e}^{jz^*} = \frac{1}{(q^{-2}\zeta;q^{-2})_{2j}}, \tag{2.27}$$

$$\mathrm{e}^{-jz^*}\,\mathrm{e}^{-jz} = (q^{-2}\zeta;q^{-2})_{2j}, \qquad \mathrm{e}^{-jz}\,\mathrm{e}^{-jz^*} = (\zeta;q^2)_{2j}, \tag{2.28}$$

where $(a;q)_n$ is the q-shifted factorial. For a positive integer n the identities given below hold:

$$(x^*)^n x^n = \frac{q^{n(n-2)}\zeta^n}{(\zeta;q^2)_n}, \qquad x^n(x^*)^n = \frac{q^{-n(n+2)}\zeta^n}{(q^{-2}\zeta;q^{-2})_n}. \tag{2.29}$$

We will use these relations subsequently. Definitions and formulae of q-analysis used in this work are summarized in the appendix.

The coproduct of $\mathcal{A}$ can be regarded as a left/right coaction of $\mathcal{A}$ on $\mathcal{A}$ itself. This leads a left and a right coaction of $\mathcal{A}$ on x and y, respectively. Denoting these coactions by $\varphi(x)$, $\varphi(y)$ we have

$$\varphi(x) = q^{-1/2}\Delta(bd^{-1}) = (a\otimes x + b\otimes q^{-1/2})(q^{1/2}c\otimes x + d\otimes 1)^{-1}, \tag{2.30}$$

$$\varphi(y) = q^{1/2}\Delta(d^{-1}c) = (q^{-1/2}y\otimes b + 1\otimes d)^{-1}(y\otimes a + q^{1/2}\otimes c). \tag{2.31}$$

It is easy to verify that (2.26) is covariant under the coaction (2.30).

3. $SU_q(2)$ coherent states

3.1. *Definition of coherent states*

The universal $\mathcal{T}$-matrix (2.15) was constructed as a quantum analogue of an exponential mapping. As in [20], the coherent state for $\mathcal{A}$ is defined as a quantum analogue of the generalized coherent state. Namely, we consider a representation of $\mathcal{U}$ and take a fiducial vector from the representation space. The coherent state is the state obtained by transformation of the fiducial vector by the universal $\mathcal{T}$-matrix. Let us take the spin j representation of $\mathcal{U}$

$$J_\pm|jm\rangle = \sqrt{[j\mp m]_q[j\pm m+1]_q}\,|jm\pm 1\rangle, \qquad J_0|jm\rangle = m|jm\rangle, \tag{3.1}$$

where j is a non-negative integer or a positive half-integer. This is a unitary representation for $q\in U(1)$ or $q\in\mathbb{R}$. We assume $q\in\mathbb{R}$ throughout this paper. The coherent state for $\mathcal{A}$ is defined by

$$|x,z\rangle = \mathcal{T}|j\ -j\rangle. \tag{3.2}$$

Repeated use of (3.1) gives us the explicit form of the coherent state

$$|x,z\rangle = \sum_{n=0}^{2j} q^{nj}\begin{bmatrix}2j\\ n\end{bmatrix}_q^{1/2} x^n\,\mathrm{e}^{-jz}|j\ -j+n\rangle \tag{3.3}$$

$$= \mathrm{e}^{-jz} \sum_{n=0}^{2j} q^{-nj} \begin{bmatrix} 2j \\ n \end{bmatrix}_q^{1/2} x^n |j - j + n\rangle. \tag{3.4}$$

In the classical case if two coherent states differ from one another only by a phase factor, they are regarded as the same state. Thus the coherent state has one-to-one correspondence to a point of the coset space $SU(2)/U(1)$. The standard choice [2, 3, 24] of the coset representative with z being real, is given by

$$\mathrm{e}^z = 1 + |x|^2, \qquad x^* = -y, \qquad y^* = -x. \tag{3.5}$$

In the quantum case, however, a choice of real z is inadmissible since, as may be seen from (2.25), it leads to a contradiction

$$(1 - q^{-2})\zeta = 0.$$

We thus take the universal $\mathcal{T}$-matrix itself as our coset representative.

By definition, the coherent state (3.2) has unit norm. Proof by direct computation is also easy. We give the proof below in order to see that the factor e^{-jz} behaves classically. With the aid of the identities (A.4) and (A.5) for q-shifted factorials, we evaluate the norm

$$\langle x, z|x, z\rangle = \mathrm{e}^{-jz^*} \mathrm{e}^{-jz} \sum_{n=0}^{2j} (-1)^n q^{n(n-1)} \frac{(q^{-4j}; q^2)_n}{(q^2; q^2)_n (q^{-4j}\zeta; q^2)_n} \zeta^n.$$

It is remarkable that $\mathrm{e}^{-jz^*}\mathrm{e}^{-jz}$ is factored out and, because of (2.28), commutes with the rest of the expression. This precisely happens in the classical case. Expressing the norm in terms of the basic hypergeometric function and using (A.8), it is observed to be equal to unity

$$\begin{aligned} \langle x, z|x, z\rangle &= \mathrm{e}^{-jz^*}\mathrm{e}^{-jz} {}_1\phi_1 \left[\begin{matrix} q^{-4j} \\ q^{-4j}\zeta \end{matrix}; q^2; \zeta \right] \\ &= \mathrm{e}^{-jz^*}\mathrm{e}^{-jz} \frac{(\zeta; q^2)_\infty}{(q^{-4j}\zeta; q^2)_\infty} = 1. \end{aligned}$$

3.2. Resolution of unity

One of the most important properties of the generalized coherent states is 'resolution of unity'. This is essential for applications of coherent states to physical problems, path integrals, representation theory, and so on. Despite the noncommutativity of $\mathcal{A}$, the coherent state (3.3) satisfies the resolution of unity with respect to an invariant integration over quantum groups. Invariant integration over the quantum group $\mathcal{A}$ has been discussed in [12, 19, 25, 26]. Attempt was made in [19] to develop a semiclassical approach towards the construction of the quantum Haar measure. In our work we follow the description given in [27].

Let $\mathcal{G}$ be an arbitrary quantum group. A linear functional $H : \mathcal{G} \longrightarrow \mathbb{C}$ is said to be *normalized bi-invariant integral* if

(1) $H[1_{\mathcal{G}}] = 1$,
(2) for any $f \in \mathcal{G}$

$$(H \otimes \mathrm{id})[\Delta(f)] = (\mathrm{id} \otimes H)[\Delta(f)] = H[f]. \tag{3.6}$$

Let $\mathcal{V}$ be an algebra dual to $\mathcal{G}$. Writing the coproduct of $f \in \mathcal{G}$ as $\Delta(f) = \sum_k f_k \otimes f^k$, the left and the right action of $Z \in \mathcal{V}$ on f are defined by

$$Z \triangleright f = \sum_k f_k \langle Z, f^k\rangle, \qquad f \triangleleft Z = \sum_k \langle Z, f_k\rangle f^k. \tag{3.7}$$

Then one can prove the left and the right invariance of the integral H

$$H[Z \triangleright f] = H[f \triangleleft Z] = \epsilon(Z)H[f]. \tag{3.8}$$

Below we cite some results for $\mathcal{G} = SL_q(2,\mathbb{C})$, $\mathcal{V} = U_q[sl(2,\mathbb{C})]$ from [27], since they are used in the proof of resolution of unity (note that our conventions are slightly different from those in [27]). From the invariance under the action of q^{J_0}, it is shown that

$$H[a^\alpha b^\beta c^\gamma d^\delta] \neq 0, \qquad \text{only if} \quad \alpha = \delta \quad \text{and} \quad \beta = \gamma, \tag{3.9}$$

$$H[b^\beta c^\gamma d^{-\delta} a^{-\alpha}] \neq 0, \qquad \text{only if} \quad \alpha = \delta \quad \text{and} \quad \beta = \gamma. \tag{3.10}$$

Implementing unit value of the quantum determinant the monomials of the form $a^\alpha b^\beta c^\beta d^\alpha$ and $b^\beta c^\beta d^{-\alpha} a^{-\alpha}$ are reduced to a power series of the element ζ defined in (2.24). Integration of ζ^n is computed using bi-invariance (3.6)

$$H[\zeta^n] = \frac{q^{2n}}{(n+1)_{q^2}}. \tag{3.11}$$

It is worth noting that that for a polynomial $f(\zeta)$ the invariant integral can be written as a q-integral

$$H[f(\zeta)] = \int_0^1 f(q^2\zeta)\, \mathrm{d}_{q^2}\zeta. \tag{3.12}$$

With these preparations, one can show that the coherent state for quantum group $\mathcal{A}$ provides the resolution of unity

$$(2j+1)_{q^2} H[|x,z\rangle\langle x,z|] = 1. \tag{3.13}$$

Proof. By virtue of (2.28), the integrand is written as

$$|x,z\rangle\langle x,z| = \sum_{n,m=0}^{2j} q^{j(n+m)} \begin{bmatrix} 2j \\ n \end{bmatrix}_q^{1/2} \begin{bmatrix} 2j \\ m \end{bmatrix}_q^{1/2} (q^{-2n}\zeta; q^2)_{2j} x^n (x^*)^m |j\ -j+n\rangle\langle j\ -j+m|. \tag{3.14}$$

Noting that $x^n(x^*)^m = (-1)^m q^{(m^2-n^2)/2-m-nm} b^n c^m d^{-n} a^{-m}$, we see with the aid of (3.10) that the terms in (3.14) contribute to the integration (3.13) only if $n = m$:

$$\begin{aligned} H[|x,z\rangle\langle x,z|] &= \sum_{n=0}^{2j} q^{2jn} \begin{bmatrix} 2j \\ n \end{bmatrix}_q H[(q^{-2n}\zeta; q^2)_{2j} x^n (x^*)^n] |j\ -j+n\rangle\langle j\ -j+n| \\ &\overset{(2.29)}{=} \sum_{n=0}^{2j} q^{2jn-n(n+2)} \begin{bmatrix} 2j \\ n \end{bmatrix}_q H[(\zeta; q^2)_{2j-n}\zeta^n] |j\ -j+n\rangle\langle j\ -j+n|. \end{aligned}$$

Before proceeding further, we need to show two identities

$$(\zeta; q)_m = \sum_{k=0}^{m} q^{mk} \frac{(q^{-m}; q)_k}{(q;q)_k} \zeta^k, \tag{3.15}$$

$$\sum_{k=0}^{m-n} \frac{q^{(m-n+1)k}}{(n+k+1)_q} \frac{(q^{-m+n}; q)_k}{(q;q)_k} = (-1)^n \frac{q^{-mn+n(n-1)/2}}{(m+1)_q} \frac{(q;q)_n}{(q^{-m};q)_n}. \tag{3.16}$$

The identity (3.15) is obtained by setting $a = q^{-m}$, $z = q^m\zeta$ in the q-binomial theorem (A.7). To prove (3.16), we start with a slight modification of (3.15)

$$\zeta^n(\zeta;q)_{m-n} = \sum_{k=0}^{m-n} q^{(m-n)k}\frac{(q^{-m+n};q)_k}{(q;q)_k}\zeta^{n+k},$$

and perform the q-integration on both sides. The rhs immediately yields

$$I_{m-n,n} \equiv \int_0^x \sum_{k=0}^{m-n} q^{(m-n)k}\frac{(q^{-m+n};q)_k}{(q;q)_k}\zeta^{n+k}\,\mathrm{d}_q\zeta = \sum_{k=0}^{m-n} q^{(m-n)k}\frac{(q^{-m+n};q)_k}{(q;q)_k}\frac{x^{n+k+1}}{(n+k+1)_q}.$$

We use integral by parts (A.13) for the lhs

$$\begin{aligned} I_{m-n,n} &= \frac{x^{n+1}}{(n+1)_q}(xq^{-1};q)_{m-n} - \frac{1}{(n+1)_q}\int_0^x \zeta^{n+1}D_q(\zeta q^{-1};q)_{m-n}\,\mathrm{d}_q\zeta \\ &= \frac{x^{n+1}}{(n+1)_q}(xq^{-1};q)_{m-n} + \frac{q^{-1}(m-n)_q}{(n+1)_q}I_{m-n-1,n+1}. \end{aligned}$$

As the first term vanishes for the intended value $x = q$, we do not keep track of it. Consequently the recurrence relation is easily solved

$$\begin{aligned} I_{m-n,n} &= q^{-(m-n)}\frac{(m-n)_q!}{(n+1)_q(n+2)_q\cdots(m)_q}I_{0,m} \\ &= (-1)^n q^{-(m-n)-mn+n(n-1)/2}\frac{(q;q)_n}{(q^{-m};q)_n}\frac{x^{m+1}}{(m+1)_q}. \end{aligned}$$

Setting $x = q$, we obtain (3.16).

Returning to the resolution of unity, one can now carry out the integration

$$\begin{aligned} H[|x,z\rangle\langle x,z|] &\overset{(3.15)}{=} \sum_{n=0}^{2j}(-1)^n q^{4jn-n(n+1)}\frac{(q^{-4j};q^2)_n}{(q^2;q^2)_n}\sum_{k=0}^{2j-n} q^{2(2j-n)k}\frac{(q^{-4j+2n};q^2)_k}{(q^2;q^2)_k} \\ &\quad\times H[\zeta^{n+k}]|j-j+n\rangle\langle j-j+n| \\ &\overset{(3.11)}{=} \sum_{n=0}^{2j}(-1)^n q^{4jn-n(n-1)}\frac{(q^{-4j};q^2)_n}{(q^2;q^2)_n}\sum_{k=0}^{2j-n}\frac{q^{2(2j-n+1)k}}{(n+k+1)_{q^2}}\frac{(q^{-4j+2n};q^2)_k}{(q^2;q^2)_k} \\ &\quad\times|j-j+n\rangle\langle j-j+n| \\ &\overset{(3.16)}{=} \frac{1}{(2j+1)_{q^2}}\sum_{n=0}^{2j}|j-j+n\rangle\langle j-j+n| = \frac{1}{(2j+1)_{q^2}}. \end{aligned}$$

We thus proved (3.13). □

3.3. Properties of the coherent state

The coherent state for the quantum group $\mathcal{A}$ allows for easy generalizations of some properties whose classical counterparts are well known [3]. The discussions presented here in conjunction with the results obtained in the preceding and the succeeding sections show that many key characteristics of the $SU(2)$ coherent state can be lifted up to the quantum group setting.

We first investigate the overlap of two coherent states. The coherent state $|x,z\rangle$ is an element in $\mathcal{A}\otimes V^{(j)}$, where $V^{(j)}$ is the representation space of spin j. According to [20], the overlap of two coherent states is defined as an object in $\mathcal{A}\otimes\mathcal{A}$. Let us introduce two independent copies of the coherent state distinguished by the subscripts: $|x_a, z_a\rangle, a = 1, 2$. Then the recipe given in [20] yields the overlap as

$$\langle x_1, z_1 \,|\, x_2, z_2\rangle = (\mathrm{e}^{-jz_1^*} \otimes \mathrm{e}^{-jz_2}) \sum_{n=0}^{2j} \begin{bmatrix} 2j \\ n \end{bmatrix}_q (x_1^* \otimes x_2)^n. \tag{3.17}$$

In (3.17) the exponential terms contained in the parenthesis commute with the factored sum.

Next we enlist the actions of the generators of the $\mathcal{U}$ algebra on the coherent state

$$J_+|x, z\rangle = \sum_{n=0}^{2j} q^{(n-1)j}\,[n]_q \begin{bmatrix} 2j \\ n \end{bmatrix}_q^{1/2} x^{n-1}\,\mathrm{e}^{-jz}|j - j + n\rangle, \tag{3.18}$$

$$J_-|x, z\rangle = \sum_{n=0}^{2j} q^{(n+1)j}\,[2j - n]_q \begin{bmatrix} 2j \\ n \end{bmatrix}_q^{1/2} x^{n+1}\,\mathrm{e}^{-jz}|j - j + n\rangle, \tag{3.19}$$

$$[J_0]_q\,|x, z\rangle = \sum_{n=0}^{2j} q^{nj}\,[-j + n]_q \begin{bmatrix} 2j \\ n \end{bmatrix}_q^{1/2} x^{n}\,\mathrm{e}^{-jz}|j - j + n\rangle. \tag{3.20}$$

It now follows that there exists an operator which annihilates the coherent state

$$(J_- + (1 + q^{2j})x\,[J_0]_q - q^{2j}x^2 J_+)|x, z\rangle = 0. \tag{3.21}$$

Furthermore, the coherent state is an eigenvector of the operator Γ defined by

$$\Gamma = (1 - (q^j + q^{-j})\zeta)q^{J_0}\,[J_0]_q - (1 - q^j\zeta)xq^{J_0}J_+ + q^{-j-1}\mathrm{e}^{-z}yq^{J_0}J_-. \tag{3.22}$$

The eigenvalue relation reads

$$\Gamma|x, z\rangle = -q^{-j}\,[j]_q\,|x, z\rangle. \tag{3.23}$$

Relations (3.21) and (3.23) may be proven by straightforward computation.

4. The coherent state and the q-sphere

Geometrical importance of coherent states is due to the fact that they provide natural description of the Kähler structure of homogeneous spaces [28]. Let us recall the case of $SU(2)$. The homogeneous space for the $SU(2)$ coherent state is $SU(2)/U(1) \approx S^2$. Expectation values of the $su(2)$ Lie algebra elements with respect to the $SU(2)$ coherent state reflect this fact

$$\langle J_+\rangle\langle J_-\rangle + \langle J_0\rangle^2 = \langle J_x\rangle^2 + \langle J_y\rangle^2 + \langle J_z\rangle^2 = j^2. \tag{4.1}$$

Since the expectation values $\langle J_a\rangle$ are functions on a complex plane, this provides a complex description of the 2-sphere: $S^2 \approx \mathbb{C} \cup \{\infty\}$. The Kähler potential on S^2 is given by the normalization factor of the coherent state: $F(x, x^*) = -\ln|\,\langle j - j\,|\,x\rangle\,|^2$.

Now we turn to the quantum group setting. Homogeneous spaces for quantum groups are introduced in [29]. Making use of the quantum subgroups of $\mathcal{A}$ consisting of the set of diagonal matrices $U(1) = \{\mathrm{diag}(\alpha, \alpha^*) || \alpha| = 1, \alpha \in \mathbb{C}\}$, one can see that the homogeneous space $SU_q(2)/U(1)$ is generated by ab, bc, cd. Thus the homogeneous space $SU_q(2)/U(1)$ is identified with the Podleś q-sphere [22] embedded into $\mathcal{A}$ [29, 30]

$$x_{-1} = \sqrt{1 + q^2}ab, \qquad x_0 = 1 + (q + q^{-1})bc, \qquad x_1 = \sqrt{1 + q^{-2}}dc. \tag{4.2}$$

These coordinates of q-sphere satisfy the relations

$$\begin{aligned}
&x_0^2 - q^{-1}x_1x_{-1} - qx_{-1}x_1 = 1,\\
&(1-q^{-2})x_0^2 + q^{-1}x_{-1}x_1 - q^{-1}x_1x_{-1} = (1-q^{-2})x_0,\\
&x_{-1}x_0 - q^{-2}x_0x_{-1} = (1-q^{-2})x_{-1},\\
&x_0x_1 - q^{-2}x_1x_0 = (1-q^{-2})x_1.
\end{aligned} \tag{4.3}$$

This is a special case of the q-sphere $S^2_{q,\rho}$ for a specific value of the parameter ρ. Infinitesimal characterization [29, 30] for this q-sphere is given by

$$(q^{J_0} - q^{-J_0}) \triangleright x_k = 0, \qquad k = \pm 1, 0. \tag{4.4}$$

Let us consider expectation values of some specific elements in $\mathcal{U}$ with respect to the coherent state for $\mathcal{A}$

$$\begin{aligned}
&X_+ \equiv \langle x, z|J_+q^{-J_0}|x, z\rangle = [2j]_q\,(1-\zeta)x^*,\\
&X_- \equiv \langle x, z|q^{-J_0}J_-|x, z\rangle = [2j]_q\, x(1-\zeta),\\
&X_0 \equiv \langle x, z|q^{-J_0}\,[J_0]_q\,|x, z\rangle = q^{-2}\,[2j]_q\,\zeta - q^j\,[j]_q\,.
\end{aligned} \tag{4.5}$$

These expectation values, after suitable scaling given by

$$\begin{aligned}
&x_1 = -q\frac{\sqrt{[2]_q}}{[2j]_q}X_+ = -q\sqrt{[2]_q}(1-\zeta)x^*,\\
&x_0 = 1 - q\frac{[2]_q}{[2j]_q}(X_0 + q^j\,[j]_q) = 1 - q^{-1}\,[2]_q\,\zeta,\\
&x_{-1} = \frac{\sqrt{[2]_q}}{[2j]_q}X_- = \sqrt{[2]_q}x(1-\zeta),
\end{aligned} \tag{4.6}$$

satisfy the defining relations (4.3) of the q-sphere, while maintaining the following $*$-involution map:

$$x_1^* = -qx_{-1}, \qquad x_0^* = x_0, \qquad x_{-1}^* = -q^{-1}x_1. \tag{4.7}$$

The relation

$$\zeta = \frac{qx^*x}{1+qx^*x} = \frac{q^3xx^*}{1+qxx^*} \tag{4.8}$$

ensures that the coordinate x_k are functions of only x and x^*. Thus the coherent state gives a natural complex description of the q-sphere. We emphasize that we did not introduce any additional assumption to obtain the above complexification. It is a direct consequence of the definitions of $\mathcal{U}$ and its dual together with the finite dimensional representations of $\mathcal{U}$. Complex description of the quantum 2-sphere has been previously considered in [31, 32]. In [31], the dressing transformation [33] is used to derive the relations between the complex coordinates x, x^* of q-sphere. It looks a bit more complex than our relation (2.26). The authors of [32] introduced b^{-1}, c^{-1} to derive the stereographic projection of the q-sphere. The resulting commutation relations of complex coordinates are rather simple. However, $b, c \in \mathcal{A}$ are not assumed to be invertible in our setting.

Next we study differential calculus on the q-sphere in our complex description. Recall that differential calculus on the q-sphere in $x_{\pm 1}, x_0$ coordinates have been extensively studied [34–36]. Our aim in this section is to develop differential calculus in the complex coordinates x, x^*. Such a differential calculus is considered in [32] based on different noncommutative coordinates from ours.

As seen in (4.2), the q-sphere can be embedded in $\mathcal{A}$. The embedding allows us to infer the differential structures on the q-sphere from the well-known [12, 37] covariant differential calculi on $\mathcal{A}$. We use the so-called left-covariant 3D calculus [12] on $\mathcal{A}$ because of the reasons advanced in [32]. Along the line in [38], we list the relations of the 3D calculus on $\mathcal{A}$ in our conventions for subsequent use. Three following elements in $\mathcal{U}$

$$\mathcal{X}_1 = \frac{q^{4J_0}-1}{q^2-1}, \qquad \mathcal{X}_0 = q^{1/2}J_+q^{J_0}, \qquad \mathcal{X}_2 = q^{-1/2}J_-q^{J_0} \tag{4.9}$$

may be regarded to span the quantum tangent space on $\mathcal{A}$, and let ω_k $(k=0,1,2)$ be the 1-forms dual to these tangent vectors, respectively. The differentials of the elements a, b, c, d, denoted sequentially as $\alpha, \beta, \gamma, \delta$, relate to ω_k as follows:

$$\alpha = a\omega_1 + b\omega_2, \qquad \beta = a\omega_0 - q^{-2}b\omega_1, \qquad \gamma = c\omega_1 + d\omega_2, \qquad \delta = c\omega_0 - q^{-2}d\omega_1. \tag{4.10}$$

Commutation relations between the coordinates and differentials read

$$\begin{aligned} &\omega_1 a = q^2 a\omega_1, \qquad \omega_1 b = q^{-2}b\omega_1, \qquad \omega_1 c = q^2 c\omega_1, \qquad \omega_1 d = q^{-2}d\omega_1, \\ &\omega_k a = qa\omega_k, \qquad \omega_k b = q^{-1}b\omega_k, \qquad \omega_k c = qc\omega_k, \qquad \omega_k d = q^{-1}d\omega_k, \end{aligned} \tag{4.11}$$

where $k=0,2$. It follows that

$$[x, \omega_k] = [x^*, \omega_k] = 0, \qquad k = 0,1,2. \tag{4.12}$$

With these settings, it is an easy exercise to show that

$$\begin{aligned} &x\,\mathrm{d}x = q^2\,\mathrm{d}x\,x, \qquad x^*\,\mathrm{d}x^* = q^{-2}\,\mathrm{d}x^*\,x^*, \\ &\mathrm{d}x\,x^* = q^{-2}f_-(x^*x)x^*\,\mathrm{d}x, \qquad \mathrm{d}x^*\,x = q^2 x f_+(x^*x)\,\mathrm{d}x^*, \end{aligned} \tag{4.13}$$

where

$$f_\pm(x^*x) = \frac{1-\zeta}{1-q^{\pm4}\zeta} = \frac{1}{1+(1-q^{\pm4})qx^*x}. \tag{4.14}$$

The nilpotency of the complex differentials follows:

$$(\mathrm{d}x)^2 = (\mathrm{d}x^*)^2 = 0. \tag{4.15}$$

To determine the commutation relation between $\mathrm{d}x$ and $\mathrm{d}x^*$, we need to calculate $\mathrm{d}f_\pm(x^*x)$. This is done with the help of identities

$$\begin{aligned} &\mathrm{d}x\,t = q^{-4}tf_-(t)\,\mathrm{d}x, \qquad \mathrm{d}x^*\,t = q^4 tf_+(t)\,\mathrm{d}x^*, \\ &x\,tf_+(t) = \frac{q^{-2}t}{1-q^2\omega t}x, \qquad x^*\,tf_-(t) = \frac{q^2 t}{1+\omega t}x^*, \end{aligned}$$

where $t = x^*x$ and $\omega = q - q^{-1}$. By induction, one can show that

$$\mathrm{d}(x^*x)^n = \frac{q}{\omega}\frac{1+\omega t}{1+qt}t^{n-1}\left\{-\left(1-\frac{q^{2n}}{(1-q^2\omega t)^n}\right)x\,\mathrm{d}x^* + \left(1-\frac{q^{-2n}}{(1+\omega t)^n}\right)x^*\,\mathrm{d}x\right\}. \tag{4.16}$$

After a lengthy computation, we derive the relation

$$\mathrm{d}x\,\mathrm{d}x^* = -\frac{f_-(x^*x)}{q^2}\frac{1-q^2\omega x^*x}{1+\omega x^*x}\,\mathrm{d}x^*\,\mathrm{d}x. \tag{4.17}$$

This completes our derivation of the differential calculus on the q-sphere subject to the complexification adopted here.

5. Coherent state representation of $U_q[su(2)]$

As an application of the resolution of unity (3.13), we discuss a representation of the algebra $\mathcal{U}$ by making use of the coherent state (3.3). Let $|c\rangle$ be an arbitrary state in the representation space of spin j

$$|c\rangle = \sum_{m=-j}^{j} c_m |jm\rangle, \qquad c_m \in \mathbb{C}.$$

Then by virtue of the resolution of unity, it follows:

$$|c\rangle = \sum_{m=-j}^{j} c_m (2j+1)_{q^2} H[|x,z\rangle\langle x,z|]|jm\rangle = (2j+1)_{q^2} H[|x,z\rangle \sum_{m=-j}^{j} c_m \langle x,z \mid jm\rangle],$$

where

$$\langle x,z \mid jm\rangle = q^{j(j+m)} \begin{bmatrix} 2j \\ j+m \end{bmatrix}_q^{1/2} \mathrm{e}^{-jz^*} (x^*)^{j+m}.$$

This implies that any state in spin j representation of $U_q[su(2)]$ can be expanded in the coherent state basis. The expansion coefficient is a polynomial in x^* with degree up to $2j$.

Let us consider the monomials

$$\Psi_m^j(x) = q^{-j(j+m)-(j-m)} \begin{bmatrix} 2j \\ j+m \end{bmatrix}_q^{1/2} x^{j+m}, \tag{5.1}$$

where $m = -j, -j+1, \ldots, j$. We shall show that these monomials span a vector space carrying a unitary representation of $\mathcal{U}$ with spin j. For arbitrary elements $f(x), g(x)$ of this vector space of the monomials we adopt the following definition of an inner product:

$$(f(x), g(x)) = (2j+1)_{q^2} H[f(x)^* \,\mathrm{e}^{-jz^*}\,\mathrm{e}^{-jz} g(x)]. \tag{5.2}$$

The monomials (5.1) form an orthonormal basis with respect to the inner product

$$\left(\Psi_{m'}^j(x), \Psi_m^j(x)\right) = \delta_{m'm}. \tag{5.3}$$

The proof of relation (5.3) is summarized below. The requirement (3.10) ensures that the rhs of (5.3) vanishes for $m' \neq m$. Consequently, we just compute the case of $m' = m$. With the aid of q-binomial theorem (A.7), one verifies that

$$\begin{aligned}(x^*)^{j+m}\,\mathrm{e}^{-jz^*}\,\mathrm{e}^{-jz} x^{j+m} &= q^{(j+m)(j+m-2)} \zeta^{j+m} \frac{(q^{-2(j-m)}\zeta; q^2)_\infty}{(\zeta; q^2)_\infty} \\ &= q^{(j+m)(j+m-2)} \sum_{k=0}^{j-m} \frac{(q^{-2(j-m)}; q^2)_k}{(q^2; q^2)_k} \zeta^{j+m+k}.\end{aligned}$$

Then the invariant integration is carried out easily

$$\begin{aligned}H[(x^*)^{j+m}\,\mathrm{e}^{-jz^*}\,\mathrm{e}^{-jz} x^{j+m}] &= q^{(j+m)(j+m-2)} \sum_{k=0}^{j-m} \frac{(q^{-2(j-m)}; q^2)_k}{(q^2; q^2)_k} \frac{q^{2(j+m+k)}}{(j+m+k+1)_{q^2}} \\ &= q^{2j(j+1)+2m(j-1)} \begin{bmatrix} 2j \\ j+m \end{bmatrix}_q^{-1} \frac{1}{(2j+1)_{q^2}},\end{aligned} \tag{5.4}$$

where the last equality is due to the identity

$$\sum_{k=0}^{j-m} \frac{(q^{-(j-m)}; q)_k}{(q; q)_k} \frac{q^{j+m+k}}{(j+m+k+1)_q} = q^{j^2-m^2} \frac{(j+m)_q!(j-m)_q!}{(2j)_q!} \frac{q^{2j}}{(2j+1)_q}. \tag{5.5}$$

Using (A.14) and (A.16), the identity (5.5) is proved in a way parallel to (3.16). Then the normalization condition $\big(\Psi^j_m(x), \Psi^j_m(x)\big) = 1$ follows immediately from (5.4).

Now we introduce a formal derivative operator ∂ by

$$\partial x^n = n x^{n-1}, \tag{5.6}$$

and employ the q-derivative to realize the quantum algebra $\mathcal{U}$ on the space spanned by $\Psi^j_m(x)$. Towards this end we define the operators

$$J_+ = ((2j)_{q^2} x - x^2 D_{q^2}) q^{3/2-3j-J_0}, \qquad J_- = D_{q^2} q^{-1/2+j-J_0}, \qquad J_0 = x\partial - j. \tag{5.7}$$

Their actions on the monomial basis set read

$$J_0 \Psi^j_m = m \Psi^j_m, \qquad J_\pm \Psi^j_m = \sqrt{(j \mp m)_{q^2} (j \pm m + 1)_{q^2}}\, \Psi^j_{m\pm1}. \tag{5.8}$$

The Hermiticity of the representation (5.8) is ensured by the relations

$$\big(J_0^* \Psi^j_{m'}, \Psi^j_m\big) = \big(J_0 \Psi^j_{m'}, \Psi^j_m\big), \qquad \big(J_\pm^* \Psi^j_{m'}, \Psi^j_m\big) = \big(J_\mp \Psi^j_{m'}, \Psi^j_m\big),$$

which may be easily proven by straightforward computation.

6. High spin limit

In this section, we study the contraction corresponding to the high spin limit. Such contraction of the algebras $\mathcal{U}$ and $\mathcal{A}$, that yields a quantum deformation of one-dimensional Heisenberg algebra and group, respectively, has been discussed in [39]. We apply the contraction of $\mathcal{U}$ to our scheme. Consider the transformation of the generators of $\mathcal{U}$

$$A^\dagger = \frac{J_+}{\sqrt{j}}, \qquad A = \frac{J_-}{\sqrt{j}}, \qquad H = \frac{2}{j} J_0, \qquad q = e^{w/j}. \tag{6.1}$$

The parameter w is assumed to be real number. Keeping w finite, we take the limit of $j \to \infty$. Then the commutation relations in (2.1) yield

$$[H, A] = [H, A^\dagger] = 0, \qquad [A, A^\dagger] = \frac{\sinh wH}{w}. \tag{6.2}$$

The Hopf algebra mappings and $*$-involution are also contracted so that we have

$$\begin{aligned} &\Delta(H) = H \otimes 1 + 1 \otimes H, \\ &\Delta(X) = X \otimes e^{wH/2} + e^{-wH/2} \otimes X, \qquad (X = A, A^\dagger), \\ &\epsilon(X) = 0, \qquad S(X) = -X, \qquad (X = H, A, A^\dagger), \end{aligned} \tag{6.3}$$

and

$$(A^\dagger)^* = A, \qquad A^* = A^\dagger, \qquad H^* = H. \tag{6.4}$$

In the classical limit of $w \to 0$ relations (6.2), (6.3) and (6.4) reduce to their counterparts for the one-dimensional Heisenberg algebra. Thus the Hopf $*$-algebra introduced above is a quantum deformation of the Heisenberg algebra, which we shall denote as $U_q[h_1]$.

We now turn to the contraction of the dual generators. The following transformations of generators of $\mathcal{A}$:

$$\tilde{x} = \sqrt{j} x, \qquad \tilde{y} = \sqrt{j} y, \qquad \tilde{z} = \frac{j}{2} z \tag{6.5}$$

preserve the dual pairing between generators of $\mathcal{U}$ and $\mathcal{A}$. Taking the limit $j \to \infty$, we obtain the commutation relations

$$[\tilde{x}, \tilde{y}] = 0, \qquad [\tilde{x}, \tilde{z}] = w\tilde{x}, \qquad [\tilde{y}, \tilde{z}] = w\tilde{y}. \tag{6.6}$$

At this stage we implement the contraction of relation (2.15) to extract the universal $\mathcal{T}$-matrix for the quantum Heisenberg algebra described above

$$\mathcal{T} = \exp(\tilde{x} \otimes \mathrm{e}^{-wH/2} A^{\dagger}) \exp(\tilde{z} \otimes H) \exp(\tilde{y} \otimes \mathrm{e}^{wH/2} A). \tag{6.7}$$

For the three-dimensional representation π of the algebra $U_q[h_1]$

$$\pi(A^{\dagger}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \pi(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \pi(A^{\dagger}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.8}$$

the universal $\mathcal{T}$-matrix (6.7) yields the matrix quantum group

$$(\mathrm{id} \otimes \pi)(\mathcal{T}) = \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.9}$$

We observe that the matrix quantum group (6.9) is identical to the quantum Heisenberg group $H_q(1)$ given in [39]. The Hopf algebra maps read [39]

$$\begin{aligned}
&\Delta(\tilde{x}) = \tilde{x} \otimes 1 + 1 \otimes \tilde{x}, \qquad \Delta(\tilde{y}) = \tilde{y} \otimes 1 + 1 \otimes \tilde{y}, \\
&\Delta(\tilde{z}) = \tilde{z} \otimes 1 + 1 \otimes \tilde{z} + \tilde{y} \otimes \tilde{x}, \\
&\epsilon(a) = 0, \qquad (a = \tilde{x}, \tilde{y}, \tilde{z}), \\
&S(\tilde{x}) = -\tilde{x}, \qquad S(\tilde{y}) = -\tilde{y}, \qquad S(\tilde{z}) = -\tilde{z} + \tilde{x}\tilde{y}.
\end{aligned} \tag{6.10}$$

The quantum group $H_q(1)$ is a Hopf $*$-algebra with the following involution map:

$$\tilde{x}^* = -\tilde{y}, \qquad \tilde{y}^* = -\tilde{x}, \qquad \tilde{x}^* = -\tilde{z} + \tilde{x}\tilde{y}. \tag{6.11}$$

It is obvious that $\mathcal{T}^*\mathcal{T} = 1 \otimes 1$.

Alternatively, one can follow the prescription of Fronsdal and Galindo. Taking the basis $E_{k\ell m} = (A^{\dagger})^k H^{\ell} A^m$ of $U_q[h_1]$, we repeat the process in section 2. The dual basis is determined to be

$$e^{k\ell m} = \frac{\tilde{x}^k}{k!} \frac{\left(\tilde{z} - \frac{1}{2}(k-m)w\right)^{\ell}}{\ell!} \frac{\tilde{y}^m}{m!}. \tag{6.12}$$

Commutation relations (6.6), the Hopf structure (6.10), and the universal $\mathcal{T}$-matrix (6.7) are recovered confirming the validity of the contraction procedure.

To construct the coherent states for $H_q(1)$ algebra, we first note its Fock space representation

$$\begin{aligned}
&A|pn\rangle = \sqrt{n \frac{\sinh wp}{w}} |pn-1\rangle, \\
&A^{\dagger}|p;n\rangle = \sqrt{(n+1)\frac{\sinh wp}{w}} |pn+1\rangle, \\
&H|p;n\rangle = p|p;n\rangle, \qquad p \in \mathbb{R}, \quad n = 0, 1, 2, \ldots.
\end{aligned} \tag{6.13}$$

Parallel to the case of the ordinary bosons, the coherent state for the $H_q(1)$ algebra is constructed on the vacuum

$$|\tilde{x}, \tilde{z}\rangle = \mathcal{T}|p0\rangle = \mathrm{e}^{p\tilde{z}} \sum_{n=0}^{\infty} \left(\frac{\mathrm{e}^{2pw} - 1}{2w} \right)^{n/2} \frac{\tilde{x}^n}{\sqrt{n!}} |pn\rangle. \tag{6.14}$$

The factor $\mathrm{e}^{p\tilde{z}}$ normalizes the state appropriately. This is verified by using the identity

$$\mathrm{e}^{p\tilde{z}^*}\,\mathrm{e}^{p\tilde{z}} = \exp\left(-\frac{\mathrm{e}^{2pw} - 1}{2w} \tilde{x}^* \tilde{x} \right).$$

It is remarkable that $[\tilde{x}^*, \tilde{x}] = 0$. Consequently, the Kähler geometry is almost trivial. However, the noncommutativity of $\tilde{z}$ and $\tilde{x}, \tilde{y}$ plays a crucial role regarding, for instance, the computations of expectation values and the resolution of unity. The resolution of unity here is much simpler than that of the $SU_q(2)$ coherent states. Noting that

$$\mathrm{e}^{p\tilde{z}}\,\mathrm{e}^{p\tilde{z}^*} = \exp\left(-\frac{1-\mathrm{e}^{-2pw}}{2w}\tilde{x}^*\tilde{x}\right),$$

the invariant integration is reduced to the usual integration on the complex plane by regarding $\tilde{x}$ as a complex variable and $\tilde{x}^*$ as its conjugate. Setting $\tilde{x} = r\,\mathrm{e}^{\mathrm{i}\theta}$, it is easy to verify that

$$\int \mathrm{d}\mu|\tilde{x}, \tilde{z}\rangle\langle\tilde{x}, \tilde{z}| = 1, \qquad \mathrm{d}\mu = \frac{1-\mathrm{e}^{-2pw}}{2\pi w} r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{6.15}$$

7. Concluding remarks

We have investigated the $SU_q(2)$ coherent states in detail. It was shown that properties analogous to the classical $SU(2)$ coherent states also hold for its quantum group counterpart. A characteristic feature of $SU_q(2)$ coherent states is known to be the noncommutativity of the variable parametrizing the states. Thanks to this fact, we obtained a natural description of the q-sphere in complex coordinates. Our description of the differential calculus on the complexified q-sphere may provide essential tools for constructing the path integrals on it. In addition, this may pave a way for studying the Kähler structure on the q-sphere. Probably one can generalize this to other noncommutative analogues of Kählerian manifolds. Furthermore, similarity of representation theory between $su(2)$ and the Lie superalgebra $osp(1/2)$ encourages us to study the coherent state for the quantum supergroup $OSp_q(1/2)$[40]. The universal $\mathcal{T}$-matrix for $OSp_q(1/2)$ has been obtained [41] and the finite dimensional representations of $OSp_q(1/2)$ are well studied [40–43]. We are ready to study coherent states for $OSp_q(1/2)$. Once the $OSp_q(1/2)$ coherent states are obtained, they will give a complex description of the q-supersphere introduced in [44]. Along lines similar to the classical case [45], one will be able to discuss the noncommutative version of the super-Kähler geometry using the $OSp_q(1/2)$ coherent states. This work is in progress.

We have made important observations such as the resolution of unity in the context of the $SU_q(2)$ coherent states. As is well known, many applications of coherent states stem from this property. We thus believe the coherent states discussed in this paper have potential for various applications in physics or mathematics where noncommutativity plays certain roles. One such possibility may be the coherent state description of operators. As shown in [46], operators appearing in the analysis of the spin system are described by spherical harmonics. It may be expected that consideration of operators consisting of the generators of $\mathcal{U}$ leads us to define noncommutative version of spherical harmonics. In another development [48] it has been observed that the effective $su_q(2)$ Hamiltonians successfully reproduce the ground state properties and the spectrum of different interacting fermion-boson dynamical nuclear systems. The bosonic part of the interactions can be effectively embedded as an appropriate q-deformation of the fermionic $su(2)$ algebra. The resolution of unity via the coherent states obtained here may be useful in obtaining suitable matrix elements in these models.

Appendix. *q*-analysis

Formulae of q-analysis used in this paper are summarized in this appendix. We follow the notations and conventions of [47].

(1) q-shifted factorial

$$(a;q)_n = \begin{cases} 1, & n=0 \\ \displaystyle\prod_{k=0}^{n-1}(1-aq^k), & n \in \mathbb{N} \end{cases} \tag{A.1}$$

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{A.2}$$

(2) Useful identities

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \qquad \frac{1}{(a;q)_n} = \frac{(aq^n;q)_\infty}{(a;q)_\infty}, \tag{A.3}$$

$$\frac{(q;q)_m}{(q;q)_{m-n}} = (-1)^n q^{mn-n(n-1)/2}(q^{-m};q)_n, \tag{A.4}$$

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = (-1)^k q^{(m+1)k}\frac{(q^{-2m};q^2)_k}{(q^2;q^2)_k}. \tag{A.5}$$

(3) Basic hypergeometric series

$$ {}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q; z\right] = \sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}\{(-1)^n q^{n(n-1)/2}\}^{1+s-r}z^n. \tag{A.6}$$

(4) q-binomial theorem

$$ {}_1\phi_0\left[\begin{matrix} a \\ - \end{matrix}; q; z\right] = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{A.7}$$

(5) Special case of ${}_1\phi_1$

$$ {}_1\phi_1\left[\begin{matrix} a \\ c \end{matrix}; q; \frac{c}{a}\right] = \frac{(c/a;q)_\infty}{(c;q)_\infty}. \tag{A.8}$$

(6) q-derivative and q-integral

$$D_q f(x) = \frac{f(x)-f(xq)}{(1-q)x}, \tag{A.9}$$

$$\int_0^x f(t)\,\mathrm{d}_q t = (1-q)x\sum_{k=0}^{\infty} f(xq^k)q^k, \tag{A.10}$$

$$\int_0^x D_q f(t)\,\mathrm{d}_q t = D_q\int_0^x f(t)\,\mathrm{d}_q t = f(x). \tag{A.11}$$

Leibniz rule

$$D_q f(x)g(x) = (D_q f(x))g(xq) + f(x)D_q g(x). \tag{A.12}$$

Integral by parts

$$\int_0^x (D_q f(t))g(tq)\, \mathrm{d}_q t = f(x)g(x) - \int_0^x f(t) D_q g(t)\, \mathrm{d}_q t, \qquad \text{(A.13)}$$

$$\int_0^x (D_q f(t))g(t)\, \mathrm{d}_q t = f(x)g(x) - \int_0^x f(tq) D_q g(t)\, \mathrm{d}_q t. \qquad \text{(A.14)}$$

Some formulae

$$D_q x^n = (n)_q x^{n-1}, \qquad D_q (xq^{-1}; q)_n = -q^{-1}(n)_q (x;q)_{n-1}, \qquad \text{(A.15)}$$

$$D_q (q^{-a+k}x; q)_a = -q^{-a+k}(a)_q (q^{-a+k+1}x; q)_{a-1}, \qquad \text{(A.16)}$$

$$\int_0^x t^n \,\mathrm{d}_q t = \frac{x^{n+1}}{(n+1)_q}. \qquad \text{(A.17)}$$

Note added in proof. After submission of our paper, we were informed that Škoda has introduced coherent states for Hopf algebras based on quantum line bundles [49]. As an example, the coherent state of the $SU_q(2)$ algebra enjoying a resolution of unity was discussed. We thank Zoran Škoda for sending us his work.

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